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Ward Identity implied recursion relation at loop level

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ABSTRACT: In this article, we discuss the Ward identity for the vector currents in pure Yang-Mills theory with a pair of external lines complexified. We classify the cancelation details among the terms in the Ward identity for both tree-level currents and one-loop level currents. According to this, we find a new recursion relation for the full amplitudes and vector currents at tree-level and one-loop level. Using the new recursion relation, we obtain an efficient technique for the calculation of tree and one-loop amplitudes in Yang-Mills theory.

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1 Introduction

At tree-level, the amplitudes of pure Yang-Mills fields can be written as rational functions of external momenta and polarization vectors in spinor forms [2–7]. Such rational functions can be analyzed in detail in algebra system. According to this, BCFW recursion relation was proposed and developed in [8–10], which has been an exiting progress on the amplitudes in pure Yang-Mills theory. This was then proved in [11] using the pole structure of the tree-level on-shell amplitudes. For the theory with massive fields [12–16], the amplitudes are also rational functions of external momenta and polarization vectors in spinor forms.

At loop-level, although the whole amplitudes are no longer rational functions in general, they can be decomposed into some basic scalar integrals with coefficients being rational functions of external spinors [17, 18]. The coefficient structures are studied in depth in [19–21]. On the other hand, the integrands of the amplitudes are rational

functions of the external spinors and integral momenta. For the N=4 planar super Yang-Mills theory, [22] gives an explicit recursive formula for the all-loop integrand of scattering amplitudes.

The amplitudes in gauge theory are constrained by gauge symmetry. This leads to Ward identity which constrains the amplitudes at all loop-level. Inspired by the BCFW momenta shift, we considered the Ward identity for tree level amplitudes with complexified momenta for a pair of external lines, and then obtained a recursion relation for the boundary terms in BCFW technique in our recent article [1]. However, in that article, we chose a particular momenta shift such that the external states of the complexified lines are independent of the complex parameter z . Then a natural question is how to obtain a recursion relation for other possible momenta shifts. Furthermore, is it possible to obtain the full amplitudes and currents from the Ward identity, and to extend the technique to one-loop amplitudes? In this article, we will give positive answers to all these questions.

In section 2, we get a new recursion relation from the complexified Ward identity at tree level. The technique does not rely on the on-shell momenta shifts, and only fewer diagrams effectively contributes to the amplitudes. Furthermore the technique is also suitable for the amplitudes with one off-shell line. In the calculation using the recursion relation, four point vertices need no consideration. In section 3, we prove the Ward identity at one-loop level with real external momenta. Importantly we classify the cancelation details among the terms from different diagrams. These cancelation details can be extend to the amplitudes with complex external momenta. Then in section 4, we present the explicit recursion relations according to the cancelation details resulted from the complexified form of the Ward identity. An example is given in section 5 to show the correctness and efficiency of our method.

2 Ward Identity Implied Recursion Relation at Tree Level

In [1], we proposed that the complexified Ward identity induces a recursion relation for the boundary terms of the complexified amplitudes. Here we explain that the Ward identity can induce a new recursion relation for the full amplitudes. Furthermore, it is also possible to realize an effective recursion relation for the amplitudes with external off-shell lines L^e . We denote such amplitudes by r -rank tensor currents $J^{\mu_1\mu_2\cdots\mu_r}$, where r is the number of the external off-shell lines. For practice in the loop-level on-shell amplitude calculation, we only need the tensor currents of rank up to 2. For convenience, we denote

1-rank and 2-rank currents as vector currents and tensor currents respectively in the following.

2.1 Recursion Relations for the Vector Currents

First, we discuss vector currents at tree-level. The proof of complexified Ward identity was given in [1]. Now we will study how to deduce the recursion relation for the full amplitudes of vector currents using Ward identity. We first contract the vector currents with the momentum of the off-shell line or on-shell momentum. Then for the off-shell lines L^e , we shift its momentum and the momentum of another on-shell line L_i^s , such that the L_i^s remain on-shell. The momenta of L_i^s and L^e are $p_i = \lambda_i \tilde{\lambda}_i$, $p_e = \lambda_e \tilde{\lambda}_e + \beta_e \tilde{\beta}_e$. There are two kinds of momentum shift $\lambda_i \rightarrow \lambda_i - z\lambda_e$, $\tilde{\lambda}_e \rightarrow \tilde{\lambda}_e + z\tilde{\lambda}_i$ and $\tilde{\lambda}_i \rightarrow \tilde{\lambda}_i - z\tilde{\lambda}_e$, $\lambda_e \rightarrow \lambda_e + z\lambda_i$. The λ_e and $\tilde{\lambda}_e$ are free to choose for convenience. Hence it is possible to get three linear independent η such that they are also linear independent with p_e .

As discussed in [1], according to the fact that the complexified Ward identity holds for arbitrary complex parameter z , we can deduce by doing a derivative over z , that for the vector currents,

$$\hat{\mathcal{A}}(z)_\mu \eta_i^\mu|_{z \rightarrow 0} = -\frac{d\hat{\mathcal{A}}(z)_\mu}{dz} \hat{p}_e^\mu|_{z \rightarrow 0}, \quad (2.1)$$

where η are defined as $\hat{p}_e = p_e + z\eta$. This equation can be used to calculate the η -component of the vector currents. For the off-shell L^e , we need three linear independent η to obtain the full current. For the on-shell L^e , the left handside in 2.1 is just the amplitude upto a constant factor under the corresponding momenta shift.

The left handside of equation 2.1 can be simplified into currents of fewer on-shell lines directly. To this end, we rewrite the three point vertices which contact the external line L^e directly as

$$V_{\mu_1 \mu_2 \mu_e}^{k_e} \equiv S_{\mu_1 \mu_2 \mu_e} + R_{\mu_1 \mu_2 \mu_e} + M_{\mu_1 \mu_2 \mu_e}, \quad (2.2)$$

where

$$\begin{aligned} S_{\mu_1 \mu_2 \mu_e} &= \frac{i}{\sqrt{2}} (\eta_{\mu_1 \mu_2} (k_1 - k_2)_{\mu_e}) \\ R_{\mu_1 \mu_2 \mu_e} &= \frac{i}{\sqrt{2}} (-2\eta_{\mu_2 \mu_e} (k_e)_{\mu_1} + 2\eta_{\mu_e \mu_1} (k_e)_{\mu_2}) \\ M_{\mu_1 \mu_2 \mu_e} &= \frac{i}{\sqrt{2}} (-\eta_{\mu_2 \mu_e} (k_1)_{\mu_1} + \eta_{\mu_e \mu_1} (k_2)_{\mu_2}). \end{aligned} \quad (2.3)$$

For convenience, by contracting the three point vertex with the momentum of L^e , we denote the new vertices in the S and M terms of 2.3 as follows.

$$\begin{aligned}
\begin{array}{c} 2 \\ \diagup \\ 1 \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} 3 \\ \diagup \\ 1 \end{array} &= \frac{i}{\sqrt{2}} \eta_{\mu_1 \mu_2} k_2^2, & \begin{array}{c} 2 \\ \diagup \\ 1 \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} 3 \\ \diagup \\ 1 \end{array} &= -\frac{i}{\sqrt{2}} \eta_{\mu_1 \mu_2} k_1^2 \\
\begin{array}{c} 2 \\ \diagup \\ 1 \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} 3 \\ \diagup \\ 1 \end{array} &= -\frac{i}{\sqrt{2}} k_{\mu_2}^2 k_{\mu_1}^3, & \begin{array}{c} 2 \\ \diagup \\ 1 \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} 3 \\ \diagup \\ 1 \end{array} &= \frac{i}{\sqrt{2}} k_{\mu_1}^1 k_{\mu_2}^3
\end{aligned} \tag{2.4}$$

Using the complexified Ward identity and the cancelation details in [1], M term can be removed before doing the derivative over z . The R term is non-vanishing only when the derivative acts directly on the vertex containing L^e . The contribution from S terms cancels among the diagrams, if the derivative is not acting on the vertices containing or next to L^e or not acting propagators that shares a vertex with L^e . In other words, in 2.1 we only need consider the derivatives acting on the vertices containing or next to the L^e or the propagators connecting L^e . We write down the non-vanishing terms in Figure 1 explicitly:

$$\begin{aligned}
W^{(a)} &= \frac{i}{\sqrt{2}} \frac{(-J_1 \cdot J_2 \eta \cdot k_e - 2J_2 \cdot k_e \eta \cdot J_1 + 2J_1 \cdot k_e \eta \cdot J_2)}{k_1^2 k_2^2} \\
W^{(b)} &= \frac{i}{\sqrt{2}} \frac{(J_1 \cdot J_2 \eta \cdot k_e - 2J_1 \cdot k_e \eta \cdot J_2 + 2J_2 \cdot k_e \eta \cdot J_1)}{k_1^2 k_2^2}
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
W^{(c)} &= \frac{-i}{2} \frac{(J_1 \cdot J_2 \eta \cdot J_3 + 2J_2 \cdot J_3 \eta \cdot J_1 - 2J_1 \cdot J_3 \eta \cdot J_2)}{k_1^2 k_2^2 k_3^2} \\
W^{(d)} &= \frac{-i}{2} \frac{(-J_1 \cdot J_2 \eta \cdot J_3 - 2J_2 \cdot J_3 \eta \cdot J_1 + 2J_1 \cdot J_3 \eta \cdot J_2)}{k_1^2 k_2^2 k_3^2}
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
W^{(e)} &= \frac{i}{\sqrt{2}} \frac{(-2J_1 \cdot J_2 \eta \cdot k_1)}{k_1^2 k_2^2} \\
W^{(f)} &= \frac{i}{\sqrt{2}} \frac{(2J_1 \cdot J_2 \eta \cdot k_1)}{k_1^2 k_2^2}.
\end{aligned} \tag{2.7}$$

Thus we have represented the vector current with vector currents of fewer external states $J_{1,2,3}$. For off-shell L^e , according to the three linear independent components of the vector currents, together with the Ward identity, we can obtain the vector currents as shown in [1]. For on-shell L^s , the component of the vector current is just the amplitude. Hence our new technique can even be used to determine the vector current or the amplitude where the on-shell lines other than L^e are of same helicity.

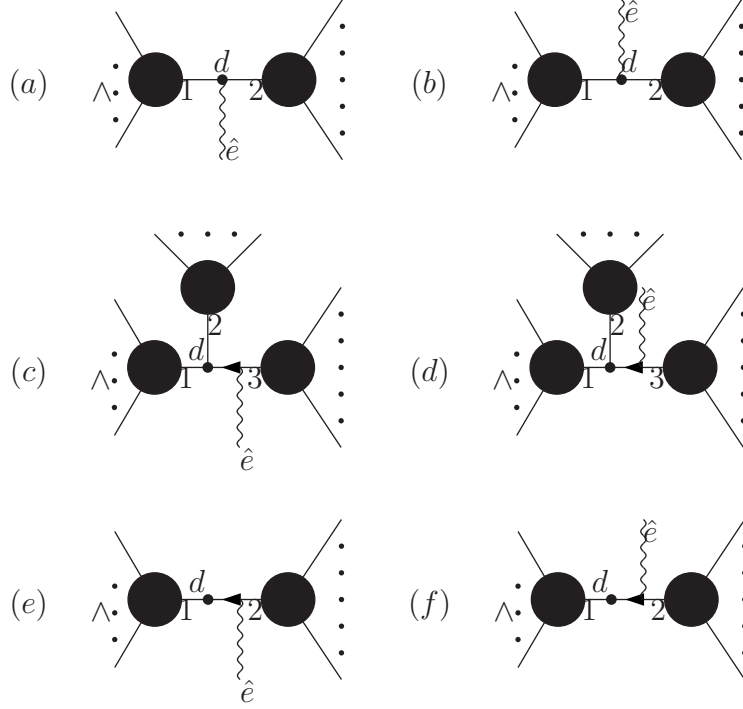


Figure 1: Effective contributions to the vector currents. Here and following, the d_\bullet denotes that we act the $\frac{d}{dz}$ on the vertices or propagators. For each three-point vertex in tree part of the diagram contacting L^e , we always remove the M terms according to the Ward identity.

2.2 Recursion Relations for Rank Two Tensor Currents

Now we discuss the tensor currents. The technique above can be generalized to the tensor currents directly. In this case, we need two independent momentum shifts for the line pairs (L_i^s, L_1^e) and (L_j^s, L_2^e) . The shifted momenta are

$$\begin{aligned} \hat{k}_i^s &= k_i^s - z\eta_1 & \hat{k}_j^s &= k_j^s - w\eta_2 \\ \hat{k}_1^e &= k_1^e + z\eta_1 & \hat{k}_2^e &= k_2^e + w\eta_2, \end{aligned} \quad (2.8)$$

where z, w are shift parameters. Then from

$$\left(\frac{d}{dz} \frac{d}{dw} \hat{A}_{\mu_1 \mu_2} \hat{k}_{e_1}^{\mu_1} \hat{k}_{e_2}^{\mu_2} \right) \Big|_{z \rightarrow 0, w \rightarrow 0} = 0 \quad (2.9)$$

we get

$$\begin{aligned} A_{\mu_1 \mu_2} \eta_1^{\mu_1} \eta_2^{\mu_2} &= - \frac{d \hat{A}_{\mu_1 \mu_2}}{dw} \Big|_{z, w \rightarrow 0} (k_1^e)^{\mu_1} \eta_2^{\mu_2} - \frac{d \hat{A}_{\mu_1 \mu_2}}{dz} \Big|_{z, w \rightarrow 0} \eta_1^{\mu_1} (k_2^e)^{\mu_2} \\ &\quad - \frac{d^2 \hat{A}_{\mu_1 \mu_2}}{dz dw} \Big|_{z, w \rightarrow 0} (k_1^e)^{\mu_1} (k_2^e)^{\mu_2}. \end{aligned} \quad (2.10)$$

The first two terms are of similar form and get contribution from Fig. 2. The third term gets contribution from Fig. 3. It is easy to see that the tensor currents can be decomposed into products of tensor currents and vector currents with fewer external on-shell lines.

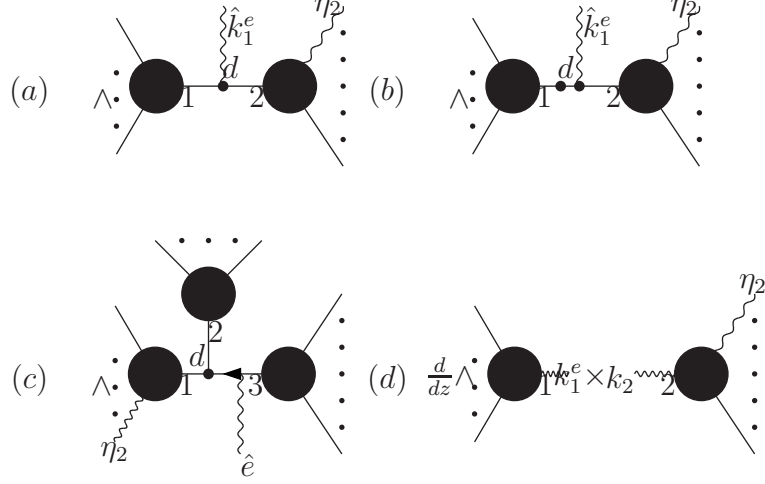


Figure 2: Effective contributions from the first term in 2.10 to the tensor currents.

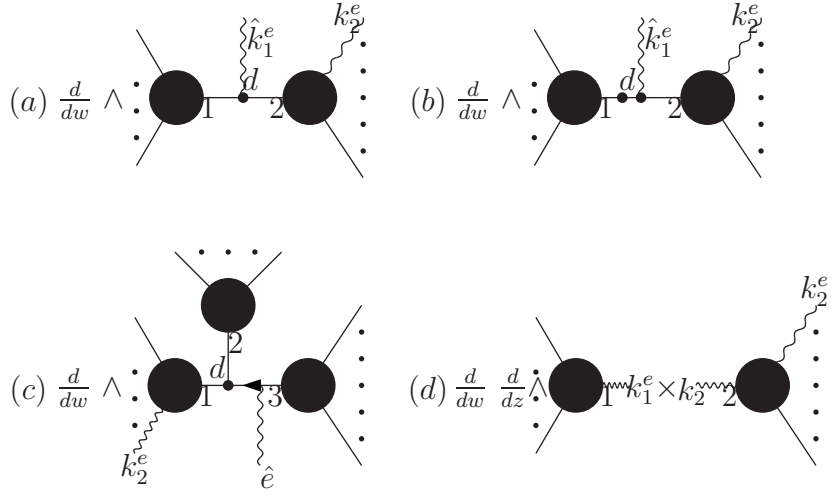


Figure 3: Effective contributions from the third term in 2.10 to the tensor currents.

3 Complexified Ward Identity at One-loop Level

At tree level, we proved that the Ward identity $A^\mu \cdot k_\mu = 0$ holds even for currents with complexified external momenta. In the proof, we discussed the cancelation in detail among the terms in $A^\mu \cdot k_\mu$. The cancelation details help simplify the calculation in practice.

In this section, we extend the technique to one-loop level for pure Yang-Mills theory. We use dimensional regularization to regularize the divergences from loop integrals. After complexifying a pair of external momenta, some lines on the loop carry complex momenta and bring ambiguous to the meaning of the loop integral. However, according to equation 2.1, what we need for our technique is the derivative of the integrand at the value $z \rightarrow 0$. And it is easy to prove that:

$$\int d^4l \frac{d}{dz} f(l^\mu, \hat{k}^\mu)|_{z \rightarrow 0} = \int d^4l \frac{d}{dz} f(-l^\mu, \hat{k}^\mu)|_{z \rightarrow 0}, \quad (3.1)$$

$$\int d^4l \frac{d}{dz} f(l^\mu, \hat{k}^\mu)|_{z \rightarrow 0} = \int d^4l \frac{d}{dz} f(l^\mu + \hat{k}'^\mu, \hat{k}^\mu)|_{z \rightarrow 0}, \quad (3.2)$$

where the $\hat{\cdot}$ over k or k' denote the dependence on z . Thus for our technique, we can translate or flip the integral variable even when the integrand is complex. We will find that the diagrams can be sorted into two sets: the contribution to $A^\mu \cdot k_\mu$ from the first set is 0 at integrand level; the contribution to $A^\mu \cdot k_\mu$ from the second set at integrand level can be related by a translation or a flip($l \rightarrow -l$) of the loop momentum, such that after integration the contribution is 0 as explained above.

Some additional attention should be paid to color orderings and symmetry factors. At tree level there is only one color ordering contributing to the the primary part of the color ordered amplitudes or currents, and the color ordered Feynman rules are given in [6]. At one loop level, most diagrams have only one color ordering except the following three kinds of diagrams: there are two three-point vertices on the YM field loop; there is a three-point vertex and a four-point vertex on the YM field loop; there are two four-point vertices on the YM field loop. For the first two cases, the contributions from the two color orderings are the same at integrand level. For the third case, the contributions from the two color orderings at integrand level differ by a translation of the integral variable, and for our technique we can view the two contributions equal as explained around equation 3.2. In a word, these three kinds of diagrams contribute a factor of 2 from possible color orderings. At the same time, these three kinds of diagrams have symmetry factor $\frac{1}{2}$, just canceling the doubling from color orderings.

We prove the Ward identity for real external momenta by induction. First we verify directly $A(\epsilon_1, \mu)k_2^\mu = 0$ and $A(\epsilon_1, \epsilon_2, \mu)k_3^\mu = 0$. The integrand of $A(\epsilon_1, \mu)k_2^\mu$ is the sum of the following terms

$$\begin{aligned}
1 \text{---} \bullet \text{---} \text{loop} \text{---} 2 &= \frac{1}{2}(d-1)(\bar{k}_{21} - k_{21}) \cdot \epsilon_1 \left(\frac{1}{k_{12}^2} \right) \\
\text{---} \bullet \text{---} \text{loop} \text{---} &= \frac{1}{2}(d-1)(\bar{k}_{21} - k_{21}) \cdot \epsilon_1 \left(-\frac{1}{\bar{k}_{12}^2} \right) \\
\text{---} \bullet \text{---} \text{loop} \text{---} &= \frac{1}{2} \frac{k_{21} \cdot \epsilon_1 k_{21} \cdot k_2}{k_{12}^2 \bar{k}_{12}^2}, & \text{---} \bullet \text{---} \text{loop} \text{---} &= \frac{1}{2} \frac{\bar{k}_{21} \cdot \epsilon_1 \bar{k}_{21} \cdot k_2}{k_{12}^2 \bar{k}_{12}^2} \\
\text{---} \bullet \text{---} \text{loop} \text{---} &= -\frac{1}{2} \frac{k_{21} \cdot \epsilon_1 \bar{k}_{12} \cdot k_2}{k_{12}^2 \bar{k}_{12}^2}, & \text{---} \bullet \text{---} \text{loop} \text{---} &= -\frac{1}{2} \frac{\bar{k}_{21} \cdot \epsilon_1 k_{12} \cdot k_2}{k_{12}^2 \bar{k}_{12}^2} \\
\text{---} \text{loop} \text{---} &= 0,
\end{aligned} \tag{3.3}$$

where k_{12} is the momentum on the loop from vertex 1 to vertex 2, and \bar{k}_{21} is the momentum on the loop from vertex 2 to vertex 1, and similarly for k_{13} etc. below.

It is easy to see that the contributions to the integrand from Δ and ∇ cancel the contribution from ghost loops (Reminder: Δ and ∇ , and other similar notations appearing in the following, refer to different parts of the three-point vertex contacting the momentum line. See the diagrams and refer to equation 2.4). The integrand from \blacktriangle and the minus of the integrand from \blacktriangledown differ by a translation of the integral variable, and the integration of the two terms adds up to 0 in sense of equation 3.2.

Then we directly verify $A(\epsilon_1, \epsilon_2, \mu)k_3^\mu = 0$. To make the cancelation more obvious, we also include the diagrams with loop inserted in one of the external on-shell lines. Such diagrams in total do not contribute to Ward identity.

The terms in the integrand of $A(\epsilon_1, \epsilon_2, \mu)k_3^\mu$ are canceled as follows:

$$\begin{aligned}
(a) \quad & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} \\
&= \frac{-1}{4\sqrt{2}} \frac{1}{k_{12}^2 k_{13}^2} [k_3 \cdot J_1 J_2 \cdot (k_{31} - k_{21}) + J_2 \cdot J_1 k_3 \cdot (k_{31} - k_{21}) \\
&\quad + (2d-4) J_2 \cdot k_3 J_1 \cdot (k_{31} - k_{21})].
\end{aligned} \tag{3.4}$$

Here d is the dimension of the spacetime. By setting $k_{21} = l - \frac{1}{2}k_1$, $k_{31} = -l - \frac{1}{2}k_1$, it is seen that the terms in (a) are all odd under $l \rightarrow -l$, and the sum will vanish after

integration.

$$\begin{aligned}
(b) \quad & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} \\
&= \frac{-1}{4\sqrt{2}} \frac{1}{k_{12}^2 k_{32}^2} [k_3 \cdot J_2 J_1 \cdot (k_{12} - k_{32}) + J_2 \cdot J_1 k_3 \cdot (k_{12} - k_{32}) \\
&\quad + (2d - 4) J_1 \cdot k_3 J_2 \cdot (k_{12} - k_{32})]
\end{aligned} \tag{3.5}$$

Similar with (a), the summation in (b) also vanishes after integration.

$$\begin{aligned}
(c) \quad & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} \\
&= \frac{-1}{4\sqrt{2}} \frac{1}{k_{13}^2 k_{32}^2} [k_3 \cdot J_2 J_1 \cdot (-k_{23}) + (-2) J_2 \cdot J_1 k_3 \cdot (-k_{23}) \\
&\quad + J_1 \cdot k_3 J_2 \cdot (-k_{23})]
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
(d) \quad & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} \\
&= \frac{-1}{4\sqrt{2}} \frac{1}{k_{13}^2 k_{32}^2} [k_3 \cdot J_2 J_1 \cdot (k_{13}) + (-2) J_2 \cdot J_1 k_3 \cdot (k_{13}) \\
&\quad + J_1 \cdot k_3 J_2 \cdot (k_{13})]
\end{aligned} \tag{3.7}$$

Thus, the summations in (c) and (d) also vanish under integration. The other summations

in the following vanish at integrand level.

$$\begin{aligned}
(e) \quad & \text{Diagram 1} + \text{Diagram 2} = 0 \\
(f) \quad & \text{Diagram 3} + \text{Diagram 4} = 0
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
(g) \quad & \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} \\
& + \text{Diagram 9} + \text{Diagram 10} + \text{Diagram 11} + \text{Diagram 12} \\
& + \text{Diagram 13} + \text{Diagram 14} + \text{Diagram 15} = 0
\end{aligned} \tag{3.9}$$

For N-point diagrams, the cancelations among different diagrams are similar with those appeared for the three point currents or amplitudes. Before we classify the cancellations, we explain some more notations: V_0 denotes the three point vertex contacting the momentum line; when V_0 is on the loop, V_1 or V_{-1} is next to V_0 on the clockwise or anti-clockwise side on the loop; $V_{\pm n}$ is the natural generalization from $V_{\pm 1}$; $D_{\pm 1}$ is the propagator between $V_{\pm 1}$ and V_0 . Then we classify the cancellations as follows:

- The contributions from \blacktriangle and \blacktriangledown part of V_0 cancel after summing over the subset of diagram as shown in the tree-level proof of Ward identity in [1].
- If the momentum line does not contact loop directly, the contributions from \triangle and ∇ cancel by inductive assumption.

- When the momentum line links to the loop directly, the contribution from Δ (and similar analysis for ∇) will lead to the momentum of D_{-1} contracting with V_{-1} . The contraction will in turn lead to four terms $\blacktriangle, \blacktriangledown, \nabla$ and Δ from the decomposition of V_{-1} . These four terms are denoted \blacktriangle_{-1} etc., and similarly for \blacktriangle_n in the following. The last term Δ_{-1} vanishes directly since the tree-level currents are conserved. The first two terms will cancel other diagrams in the same way as Case 1. Hence only the one term from ∇ is left. Then the momentum contraction continues in the same way as in the beginning of this case until it meets the three point vertex V_1 . After contracting with V_1 , both terms from \blacktriangledown_1 and ∇_1 are left. If at the beginning of this case we started with ∇ , \blacktriangle_{-1} and Δ_{-1} will be left. The four terms from $\nabla_0 \nabla_{-1} \nabla_{-2} \cdots \nabla_1$, $\nabla_0 \nabla_{-1} \nabla_{-2} \cdots \blacktriangledown_1$, $\Delta_0 \Delta_1 \Delta_2 \cdots \Delta_{-1}$ and $\Delta_0 \Delta_1 \Delta_2 \cdots \blacktriangle_{-1}$ will add up to cancel the ghost loop diagrams similarly to the situation shown in (g) of Equation 3.9 for the three point case.

Hence we have proven that Ward identity holds for any N point currents at one-loop level.

4 Implied Recursion Relations

We stated in section 2 that, at tree level to get non-vanishing contribution to the components of the vector currents, the derivative $\frac{d}{dz}$ should act on the vertices V_0 or next to V_0 (V_0 denotes the three point vertex contacting the momentum line), or act on the propagators connecting V_0 . According to the cancelation details at loop level in the previous section, the rules for acting $\frac{d}{dz}$ are modified at loop level compared to tree level rules, as follows.

At one-loop level, the action of $\frac{d}{dz}$ can be classified as:

1. When the momenta shift affect the momenta in the loop, $\frac{d}{dz}$ acts on the vertices and propagators with z -dependence in the tree part of the Feynman diagrams containing the momentum line L^e .
2. When the momenta shift do not affect the momenta in the loop, $\frac{d}{dz}$ acts on the vertices and propagators in the tree part of the Feynman diagrams containing both L^e and the other complexified external on-shell line L^s .

3. When the momenta shift affect the momenta in the loop, $\frac{d}{dz}$ acts on the vertices and propagators with z -dependence in the tree part of the Feynman diagrams containing L^s .
4. $\frac{d}{dz}$ acts on the vertices and propagators on the complex loop lines.

The first, second and third cases are similar with the tree level cases and we roughly repeat some key points. We rewrite the three point vertices V_0 as in 2.2. The M terms which are proportional to $(k_1)_{\mu_1}$ and $(k_2)_{\mu_2}$ can be removed according to the complexified Ward identity at tree and loop level. Furthermore, when $\frac{d}{dz}$ does not act on the V_0 , the R terms $\eta_{\mu_2\mu_{k_e}}(-2k_e)_{\mu_1} + \eta_{\mu_{k_e}\mu_1}(2k_e)_{\mu_2}$ do not contribute after contracting with k_e . Then we only need to consider the S terms $\frac{i}{\sqrt{2}}(\eta_{\mu_1\mu_2}(k_1 - k_2)_{\mu_{k_e}})$. If the derivative does not act on the vertices V_0 or next to V_0 , nor act on the propagators connecting V_0 , then S terms also vanish after summing over all kinds of diagrams. Above all, the total contribution from the second case will vanish, and for the first and third cases the final effective terms in the currents or amplitudes are the same as those in the tree level cases.

In the fourth case, L^e connects to the loop. According to 3.4-3.7, the diagrams with gluon loops cancel unless the derivative acts on the vertices V_0 or next to the V_0 or the propagators connecting V_0 . Then the terms proportional to $(k_1)_{\mu_1}$ and $(k_2)_{\mu_2}$ in M part of the vertex cancel the terms from corresponding diagrams with ghost loop. According to 3.8, the non-trivial contribution can be induced by acting the derivative on the vertices and propagators between L^e and the next external line on the left or right side of L^e . Hence the remaining terms after the cancelation with ghost loop diagrams induce extra contributions for the third case. Other terms will lead to same effects as those at the tree level.

First, we calculate the general formula of each diagram in Fig. 4 for the first case. Each non-vanishing contribution can be written as following.

When the derivative acts on the vertices or propagators in tree sub-diagrams, the formulae are similar with pure tree level amplitudes:

$$\begin{aligned}
W_{loop}^{(a)} &= \frac{i}{\sqrt{2}} \frac{(-J_1^{loop} \cdot J_2 \eta \cdot k_e - 2J_2 \cdot k_e \eta \cdot J_1^{loop} + 2J_1^{loop} \cdot k_e \eta \cdot J_2)}{k_1^2 k_2^2} \\
W_{loop}^{(b)} &= \frac{i}{\sqrt{2}} \frac{(J_1^{loop} \cdot J_2 \eta \cdot k_e - 2J_1^{loop} \cdot k_e \eta \cdot J_2 + 2J_2 \cdot k_e \eta \cdot J_1^{loop})}{k_1^2 k_2^2}
\end{aligned} \tag{4.1}$$

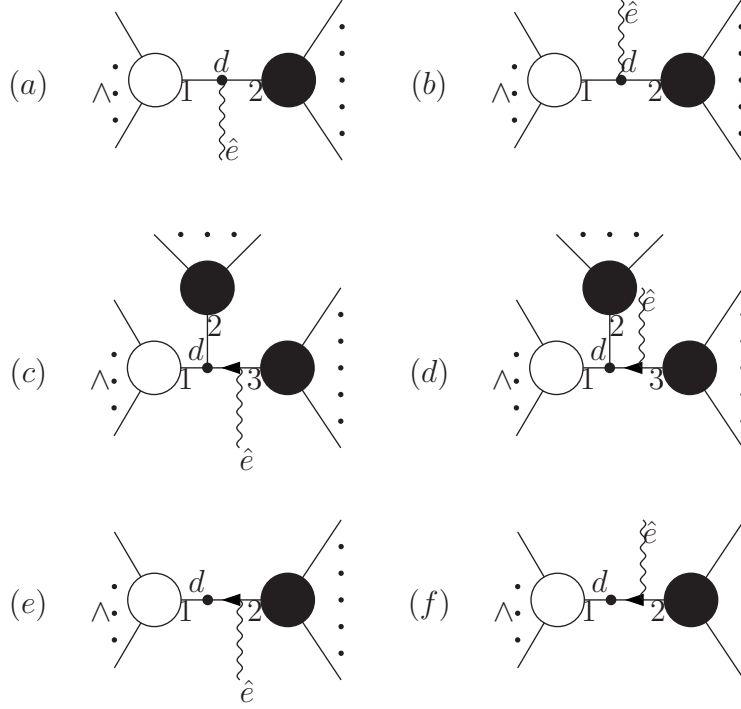


Figure 4: Effective contributions to the one-loop vector currents in Case 1. The circles and solid circles denote loop parts and tree parts of the diagrams respectively.

$$\begin{aligned}
W_{loop}^{(c)} &= \frac{-i}{2} \frac{(J_1^{loop} \cdot J_2 \eta \cdot J_3 + 2J_2 \cdot J_3 \eta \cdot J_1^{loop} - 2J_1^{loop} \cdot J_3 \eta \cdot J_2)}{k_1^2 k_2^2 k_3^2} \\
W_{loop}^{(d)} &= \frac{-i}{2} \frac{(-J_1^{loop} \cdot J_2 \eta \cdot J_3 - 2J_2 \cdot J_3 \eta \cdot J_1^{loop} + 2J_1^{loop} \cdot J_3 \eta \cdot J_2)}{k_1^2 k_2^2 k_3^2}
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
W_{loop}^{(e)} &= \frac{i}{\sqrt{2}} \frac{(-2J_1^{loop} \cdot J_2 \eta \cdot k_1)}{k_1^2 k_2^2} \\
W_{loop}^{(f)} &= \frac{i}{\sqrt{2}} \frac{(2J_1^{loop} \cdot J_2 \eta \cdot k_1)}{k_1^2 k_2^2}.
\end{aligned} \tag{4.3}$$

According to the above formulae, we find that the vector currents or amplitudes have been reduced to those with fewer external on-shell lines.

As shown in Fig. 5, when the derivative acts on the vertices on the loop, the formulae

are a little different:

$$\begin{aligned}
W_{loop}^{(c)} &= \frac{-i}{2} \frac{(J_1^{\mu\nu} g_{\mu\nu} \eta \cdot J_3 + 2J_1^{\mu\nu} (J_3)_\nu \eta_\mu - 2J_1^{\mu\nu} (J_3)_\mu \eta_\nu)}{k_1^2 l^2 (l + k_1)^2} \\
W_{loop}^{(d)} &= \frac{-i}{2} \frac{(-J_1^{\mu\nu} g_{\mu\nu} \eta \cdot J_3 - 2J_1^{\mu\nu} (J_3)_\nu \eta_\mu + 2J_1^{\mu\nu} (J_3)_\mu \eta_\nu)}{k_1^2 l^2 (l + k_1)^2}
\end{aligned} \tag{4.4}$$

The new tensor currents (with two indices) with loop momentum l can be further obtained by the method in Section 2.2.

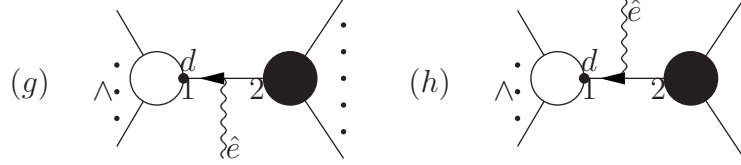


Figure 5: Effective contributions to the one-loop vector currents in Case 1 and 4. The L^e is in the tree part of the diagrams.

The second and third cases are similar.

For the fourth case, beside the contributions from diagrams in Fig. 5, the extra diagrams which will lead to non-trivial contributions are shown in Fig. 6 and Fig. 7. We can write down the total contributions to $A \cdot \eta$ from Fig. 6 directly:

$$\begin{aligned}
& \frac{-1}{(\sqrt{2})^{n+1}} \frac{d}{dz} \left(\frac{2\hat{k}_{1e} \cdot J_1 k_{en} \cdot J_n (\hat{k}_{e1} - k_{en}) \cdot k_e}{\hat{k}_{1e}^2 k_{ne}^2} + \frac{k_e \cdot J_1 k_{en} \cdot J_n}{k_{ne}^2} - \frac{\hat{k}_{1e} \cdot J_1 J_n \cdot k_4}{\hat{k}_{1e}^2} \right. \\
& \quad \left. - \frac{(k_{ne} - \hat{k}_{1e}) \cdot J_n \hat{k}_{1e} \cdot J_1}{\hat{k}_{1e}^2} + \frac{(k_{ne} - \hat{k}_{1e}) \cdot J_1 \hat{k}_{en} \cdot J_n}{\hat{k}_{1e}^2} \right) \\
& \times \frac{\hat{k}_{21} \cdot J_2 \hat{k}_{32} \cdot J_3 \cdots \hat{k}_{ii-1} \cdot \hat{J}_i k_{i+1i} \cdot J_{i+1} \cdots k_{(n-1)(n-2)} \cdot J_{n-1}}{\hat{k}_{12}^2 \hat{k}_{23}^2 \cdots \hat{k}_{(i-1)i}^2 k_{(i+1)i}^2 \cdots k_{(n-2)(n-1)}^2} \Big|_{z \rightarrow 0}
\end{aligned} \tag{4.5}$$

Here $\frac{d}{dz}$ only acts on the terms in the parentheses, since the summation of the terms in the parenthesis will vanish when $z \rightarrow 0$ if the $\frac{d}{dz}$ acts on other terms.

The terms from Fig. 7 are

$$W_{loop}^a = \frac{i}{\sqrt{2}} \frac{(-2J_{1n}^{\mu\nu} g_{\mu\nu} \eta \cdot k_1)}{k_{1e}^2 k_{ne}^2}, \tag{4.6}$$

$$W_{loop}^b = \frac{-i}{2} \frac{(J_{2n}^{\mu\nu} (J_1)_\mu \eta_\nu + 2J_{2n}^{\mu\nu} g_{\mu\nu} \eta \cdot J_1 - 2J_{2n}^{\mu\nu} (J_1)_\nu \eta_\mu)}{k_1^2 k_{1e}^2 k_{ne}^2}, \tag{4.7}$$

$$W_{loop}^c = \frac{i}{\sqrt{2}} \frac{(-J_{1n}^{\mu\nu} g_{\mu\nu} \eta \cdot k_e - 2J_{1n}^{\mu\nu} (k_e)_\nu \eta_\mu + 2J_{1n}^{\mu\nu} g_{\mu\nu} (k_e)_\mu \eta_\nu)}{k_{1e}^2 k_{ne}^2}. \quad (4.8)$$

Hence the calculation of loop diagrams is simplified to the calculation of products of tensor currents and vector currents, which can be obtained reductively by tree level technique in Section 2.

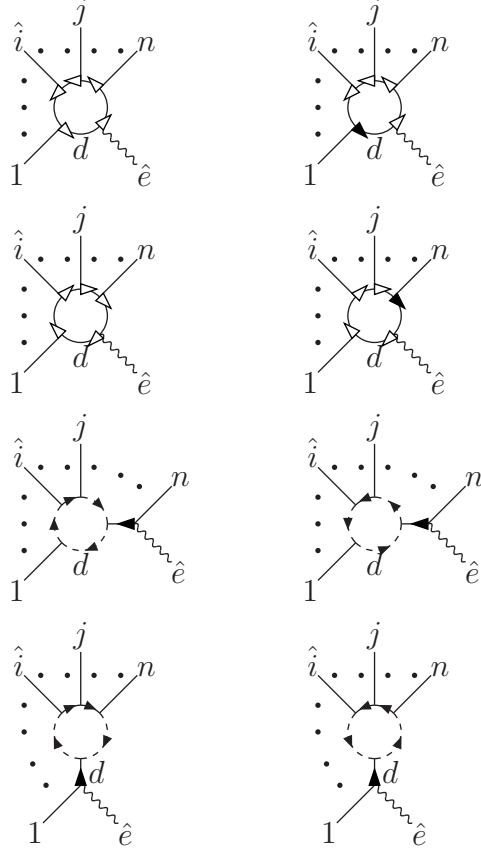


Figure 6: Diagrams with the M terms and ghost contributions. Here, L^e connect to the loop.

5 Example

As an application and verification of our technique, we compute a well-known and simple example $A(1^+, 2^+, 3^+, 4^-)$. We choose the on-shell state of line 4 to be replaced by its momentum. Since the amplitude is gauge invariant, for convenience, we can set the

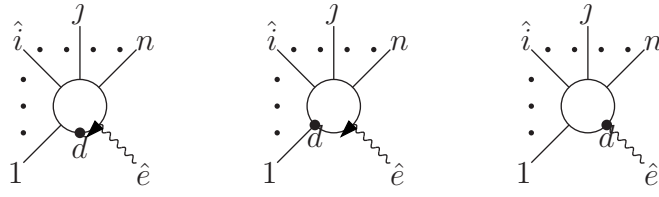


Figure 7: Diagrams with the S and R term contributions. Here, L^e connect to the loop.

reference spinors of external on-shell states as follows:

$$\epsilon_1^+ = \frac{\mu \tilde{\lambda}_1}{\langle \mu, \lambda_1 \rangle}, \quad \epsilon_2^+ = \frac{\mu \tilde{\lambda}_2}{\langle \mu, \lambda_2 \rangle}, \quad \epsilon_3^+ = \frac{\mu \tilde{\lambda}_3}{\langle \mu, \lambda_3 \rangle}, \quad \epsilon_4^- = \frac{\lambda_e \tilde{\mu}}{[\tilde{\lambda}_e, \tilde{\mu}]}.$$

We take $\mu = \lambda_e, \tilde{\mu} = \tilde{\lambda}_e$ and $k_e = k_4 = \lambda_e \tilde{\lambda}_e$. We choose complexifying lines to be 1^+ and 4^- and the momenta shifts to be:

$$\hat{k}_1 = \lambda_1 \tilde{\lambda}_1 - z \lambda_e \tilde{\lambda}_1, \quad \hat{k}_e = \lambda_e \tilde{\lambda}_e + z \lambda_e \tilde{\lambda}_1. \quad (5.1)$$

According to 2.1, the amplitude can be rewritten as

$$A(1^+, 2^+, 3^+, 4^-) = \frac{1}{[\tilde{\lambda}_e, \tilde{\lambda}_1]} \mathcal{A}_\mu \eta_i^\mu = -\frac{1}{[\tilde{\lambda}_e, \tilde{\lambda}_1]} \frac{d\hat{\mathcal{A}}(z)_\mu}{dz} \hat{p}_e^\mu|_{z \rightarrow 0}. \quad (5.2)$$

Then according to the discussion above, the possible non-vanishing contributions come from the diagrams in Fig. 8. Furthermore, according to the spinor helicity technique, the terms from Fig. (b), (c),(e),(f),(i),(k), (l), (n), (o) are all equal to zero at the integrand level and terms from Fig. (g),(h), (j), (m) vanish after integration. Hence the non-trivial terms come from (a), (d) and the terms remain from ghost cancelation as in Fig. 6:

$$\begin{aligned} A^{(a)} &\propto k_3 \cdot \epsilon_1 k_3 \cdot \epsilon_1 k_1 \cdot \epsilon_2 k_1 \cdot \epsilon_3 (-16(d-4) - 36)(D_{331} - D_{333} + D_{1133} - D_{1333}) \\ A^{(d)} &\propto k_3 \cdot \epsilon_1 k_3 \cdot \epsilon_1 k_1 \cdot \epsilon_2 k_1 \cdot \epsilon_3 (16(d-4) + 32)(C_{211} - C_{221} + C_{12} - C_{22}) \\ A^{(ghost)} &\propto k_3 \cdot \epsilon_1 k_3 \cdot \epsilon_1 k_1 \cdot \epsilon_2 k_1 \cdot \epsilon_3 4(D_{331} - D_{333} + D_{1133} - D_{1333}), \end{aligned} \quad (5.3)$$

where d is the dimension of the spacetime and the constant D, C are defined in [24]. They are constants from the loop integration. In total, the amplitude $A(1^+, 2^+, 3^+, 4^-)$ is

$$\frac{-1}{48\pi^2} \frac{k_1 \cdot k_4 + k_3 \cdot k_4}{(k_3 \cdot k_4)^2 k_1 \cdot k_4} (k_3 \cdot \epsilon_1 k_3 \cdot \epsilon_1 k_1 \cdot \epsilon_2 k_1 \cdot \epsilon_3), \quad (5.4)$$

agreeing with [25, 26].

6 Conclusion

We have discussed the Ward identity in detail for the currents and amplitudes in pure Yang-Mills theory. We find that the Ward identity with two external lines complexified holds at tree and one loop level. Then we use the Ward identity to deduce a new recursion relation for the total amplitude at tree and one loop level. In this technique, three steps are important to simplify the calculation. First, according to the complexified Ward identity, we can convert the calculation of the amplitudes or the currents to the calculation of derivative of the currents contracting with momentum. Second, we rewrite the three point vertex, the one contacting the momentum line in Ward identity, into a new form with three terms, such that when the vertex is on the loop only two terms are left, and when the vertex contact internal tree lines only one term is left. Thirdly, according to the cancelation details in the proof of complexified Ward identity, we find most terms from different diagrams cancel with each other. The number of remaining effective terms or diagrams are reduced greatly.

Comparing with the technique in our previous work [1], we find the technique in this article is more universal. Here we can obtain a recursion relation for the total amplitudes and do not need to use BCFW recursion relation. Furthermore, to use this technique, we do not need to avoid the unphysical poles and the complexified on shell states can also depend on z . Hence this technique works well for the amplitudes with any helicity structure and the momenta shifts are more general than the ones in [1]. In addition, this technique can be used for calculating one loop amplitudes with any helicity structure.

In principle, it is possible to generalize our method to higher loop cases and to other theories such as QCD. The only obstruct is to classify all the cancelation details for the Ward identity with complexified external momenta. We leave this to future work. Another extension is to combining our technique with other methods, such as unitary cut, generalized unitary cut, BCFW, OPP [27] et al to further simplify the calculation in pure Yang-Mills theory.

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References

- [1] G. Chen, *Ward Identity implied recursion relations in Yang-Mills theory*, [arXiv: 1203.6281].
- [2] S. J. Parke and T. R. Taylor, *An Amplitude for n Gluon Scattering*, Phys. Rev. Lett. **56**, 2459 (1986).
- [3] Z. Xu, D. H. Zhang and L. Chang, *Helicity Amplitudes for Multiple Bremsstrahlung in Massless Nonabelian Gauge Theories*, Nucl. Phys. B **291**, 392 (1987).
- [4] F. A. Berends and W. T. Giele, *Recursive Calculations for Processes with n Gluons*, Nucl. Phys. B **306**, 759 (1988).
- [5] D. A. Kosower, Nucl. Phys. B **335**:23 (1990).
- [6] L. J. Dixon, *Calculating scattering amplitudes efficiently*, Boulder TASI **95**, 539-584, [arXiv: hep-ph/9601359].
- [7] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, Commun. Math. Phys. **252**, 189 (2004), [arXiv: hep-th/0312171].
- [8] R. Britto, F. Cachazo and B. Feng, *Computing one-loop amplitudes from the holomorphic anomaly of unitarity cuts*, Phys. Rev. D **71**, 025012 (2005) [arXiv:hep-th/0410179].
- [9] R. Britto, F. Cachazo and B. Feng, *Generalized unitarity and one-loop amplitudes in $N = 4$ super-Yang-Mills*, Nucl. Phys. B **725**, 275-305 (2005), [arXiv:hep-th/0412103].
- [10] R. Britto, F. Cachazo and B. Feng, *New recursion relations for tree amplitudes of gluons*, Nucl. Phys. B **715**, 499-522 (2005), [arXiv:hep-th/0412308].
- [11] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, Phys. Rev. Lett. **94**, 181602 (2005), [arXiv:hep-th/0501052].
- [12] S. D. Badger, E. W. Glover, V. V. Khoze and P. Svrcek, *Recursion relations for gauge theory amplitudes with massive particles*, JHEP **0507**, 025 (2005), [arXiv: hep-th/0504159].
- [13] K. J. Ozeren and W. J. Stirling, *Scattering amplitudes with massive fermions using BCFW recursion*, Eur. Phys. J. C **48**, 159-168 (2006), [arXiv: hep-ph/0603071].
- [14] C. Schwinn, *Twistor-inspired construction of massive quark amplitudes*, Phys. Rev. D **78**, 085030 (2008), [arXiv: 0809.1442].

- [15] G. Chen, K. G. Savvidy, *Spinor formalism for massive fields with half-integral spin*, Eur. Phys. J. C **72**, 3, 1952 (2012), [arXiv:1105.3851].
- [16] G. Chen, *Recursion relations for the general tree-level amplitudes in QCD with massive dirac fields*, Phys.Rev. D83 (2011) 125005, [arXiv: 1103.2518].
- [17] Z. Bern, L. J. Dixon, David C. Dunbar and D. A. Kosower, *One loop n point gauge theory amplitudes, unitarity and collinear limits*, Nucl. Phys. B **425**, 217 (1994) , [arXiv: hep-ph/9403226].
- [18] Z. Bern, L. J. Dixon, David C. Dunbar and D. A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, Nucl. Phys. B **435**, 59 (1995), [arXiv: hep-th/9409265].
- [19] Z. Bern, V. Del Duca, L. J. Dixon, and D. A. Kosower, *All non-maximally-helicity-violating one-loop seven-gluon amplitudes in N=4 super-Yang-Mills theory*, Phys. Rev. D **71**, 045006 (2005), [arXiv: hep-th/0410224].
- [20] Z. Bern, N. E. J. Bjerrum-Bohr, David C. Dunbar and Harald Ita, *Recursive calculation of one loop QCD integral coefficients*, JHEP 0511 (2005) 027, [arXiv: hep-th/0507019].
- [21] Carola F. Berger, Zvi Bern, Lance J. Dixon, Darren Forde, David A. Kosower, *Bootstrapping One-Loop QCD Amplitudes with General Helicities.*, Phys.Rev. D74 (2006) 036009 , [arXiv: hep-ph/0604195].
- [22] Nima Arkani-Hamed, Jacob L. Bourjaily, Freddy Cachazo, Simon Caron-Huot, Jaroslav Trnka, *The All-Loop Integrand For Scattering Amplitudes in Planar N=4 SYM*, JHEP 1101 (2011) 041 , [arXiv: 1008.2958].
- [23] B. Feng, Z.B. Zhang, *Boundary Contributions Using Fermion Pair Deformation*, JHEP 1112 (2011) 057, [arXiv: 1109.1887].
- [24] R. K. Ellis, Z. Kunszt, K. Melnikov, G. Zanderighi, *One-loop calculations in quantum field theory: from Feynman diagrams to unitarity cuts*, [arXiv: 1105.4319].
- [25] Z. Bern, *Sting-based perturbative methods for gauge theories*, [arXiv: hep-ph/9304249].
- [26] S. Kharel, G. Siopsis, *Gauge theory one-loop amplitudes and the BCFW recursion relation*, [arXiv: 1111.5278].
- [27] G. Ossola, C. G. Papadopoulos, and R. Pittau, *Reducing full one-loop amplitudes to scalar integrals at the integrand level*, Nucl. Phys. B **763**, 147-169 (2007), [arXiv:hep-ph/0609007].

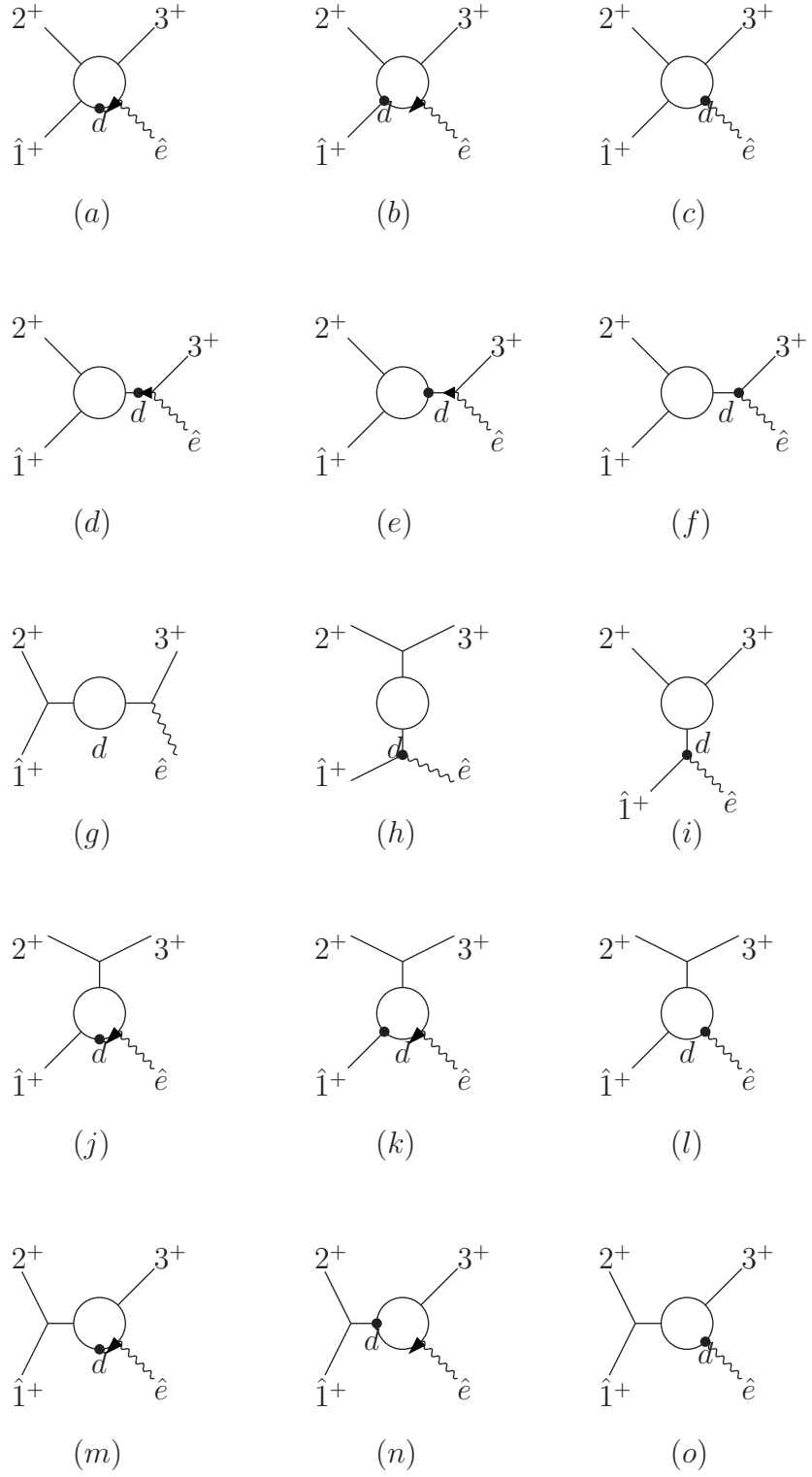


Figure 8: Effective diagrams for the four-point one-loop amplitude $A(1^+, 2^+, 3^+, 4^-)$.